TES ON FINITE DIFFERENCES.

FOR THE USE OF STUDENTS

OF

THE INSTITUTE OF ACTUARIES.

BY

A. W. SUNDERLAND, M.A.,

OF TRINITY COLLEGE, CAMBRIDGE,

AND FELLOW OF THE INSTITUTE OF ACTUARIES.

ASSISTANT ACTUARY OF THE NATIONAL LIFE ASSURANCE SOCIETY.

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CHARLES AND EDWIN LAYTON,

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PREFACE.

My object in publishing these Notes on Finite Differences is to give, in a convenient form, a collection of those elementary propositions a knowledge of which is required of the students who present themselves for the Institute of Actuaries' First Examination. I have not attempted a complete account of even that small portion of the subject utilized in actuarial science, thinking it desirable to confine myself almost entirely to the methods of elementary algebra. The only instances in which a knowledge of mathematics beyond the Binomial Theorem is required of the reader are § 2, Chap. IV; Example 28; and the Note to Example 27. These are marked with asterisks, thus *.

The difference symbol has been denoted by δ when the increment of the independent variable is taken to be unity, and by Δ when this increment is not so restricted.

I am indebted to Mr. T. B. Sprague, President of the Institute of Actuaries, for some valuable suggestions, and regret that I have been able only in part to avail myself of them.

A. W. S.

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London, E.C.

20 *Feb.* 1885.

PRELIMINARY.

§ 1. As the following short account of some of the more elementary theorems of Finite Differences is intended for the use of students who have no acquaintance with the methods of analytical geometry, it seems desirable to preface it by a short explanation of the method of representing variable quantities by curved lines or diagrams.

§ 2. Let us consider the quantity x^2. Its value, of course, depends on that of x, and if any particular value be given to x, the corresponding value of x^2 may be determined by multiplication; *e.g.*, if $x=3$, $x^2=9$.

Any quantity such as x^2 which depends on x in such a manner that for each value assigned to x it takes a determinate value is called a function of x.

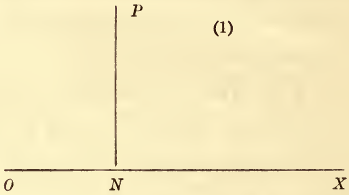
As another illustration of a function, consider, out of 100,000 persons born alive, the number, l_x, who would be living at age x on the assumption that these persons were subject to some definite law of mortality. Here for each value of x, the age attained, we have a definite number alive at that age. The quantity l_x is therefore, according to our definition, a function of x.

Any function of x may be conveniently denoted by the symbol u_x. With reference to u_x, the quantity x, (to which we may assign any value we please, the corresponding value of u_x being then determined), is called the independent variable. With reference to x, the quantity u_x, (whose value we consider determined when that of x is given), is called the dependent variable.

§ 3. Suppose we wish to examine the series of values which a function takes for a series of different values of the variable x which it involves. A very valuable and powerful method of conducting such an examination, and of investigating the properties of any function, is that of representing it by a diagram as follows:—

Let a straight line, which we will call OX, be drawn on a sheet of paper from a definite point O, and let lines measured along OX represent, according to some given scale, values of x. For example, if we take one inch to represent the unit of x, $x=5$ will be represented by measuring along OX a line, ON, 5 inches long.

In order to represent the value of the function for any value of x, draw through the end of the line which represents the value of x a line perpendicular to OX, and of a length corresponding on the same, or, it may be, a different scale to the magnitude of the function for this value of x. For example, if the function is x^2, and the scale in each case one to the inch, for $x=2$, and therefore $x^2=4$, a line ON 2 inches long must be measured along OX and through the end of it, N, a line NP drawn perpendicular to OX and 4 inches long.

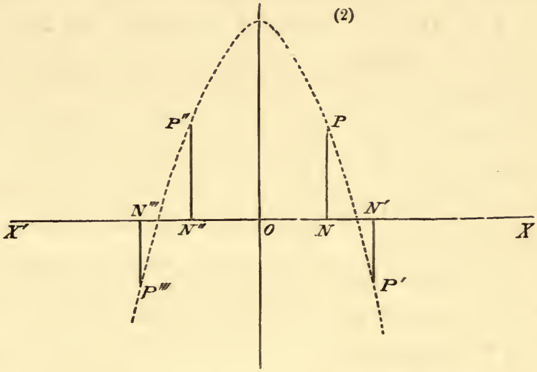


In this way, when the algebraic expression for the function is given, any number of lines such as NP may be drawn, each representing the value of the function for that value of x which is represented by the line between O and the foot of the perpendicular. The summits of these perpendiculars will form a series of points lying on a curved line, and the process above described is that of mapping out the diagram or curved line corresponding to the function.

§ 4. To complete the representation of any function of x it is necessary to provide for negative values. This is done by producing XO through O and making the convention that negative values of x shall be represented by lines measured along the produced line, which we will call OX', and negative values of the function by drawing the perpendiculars down instead of up.

It will be a useful exercise for the student to map out the diagrams for two or three different functions.

Take, for example, the function $4-x^2$. The diagram will be found to take the form shown by the dotted curved line given in Fig. (2). The curve is that known as a parabola. For the point P, x and the function are both positive;



for P' x is positive and the function negative, for P'' x is negative and the function positive, for P''' both are negative.

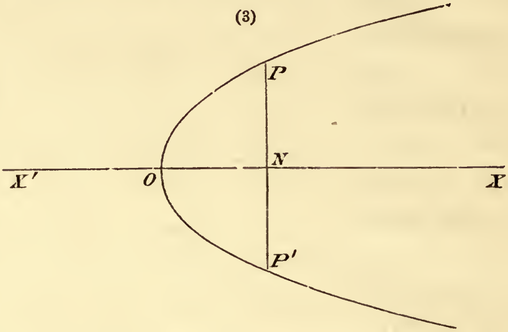
Lines such as $O\,N$ measured along $O\,X$ or $O\,X'$ are usually called abscissæ, and the perpendiculars, such as $P\,N$, ordinates.

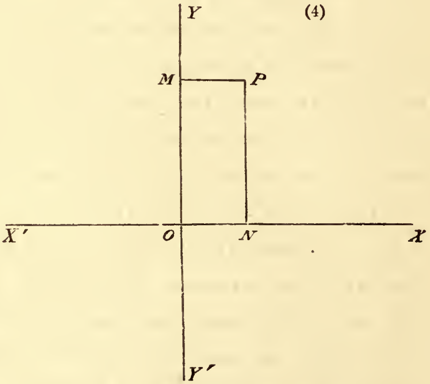
§ 5. Suppose we are given, not the algebraic expression for a function, but the curved line or diagram which forms its complete representation. Then, in order to find its value for any given value of x, we must measure along $O\,X$ a line, say $O\,N$, representing x in magnitude, and draw through N a line perpendicular to $O\,N$, so as to cut the curve. If P be the point of intersection of the perpendicular and the curve, then $N\,P$ will represent the value of the function corresponding to the given value of x. If this perpendicular, which may be drawn both up and down, meets the curve in more than one point, there will be more than one value of the function for the given value of x. If, on the other hand, it does not meet the curve at all, there will be no value of the function for the given value of x.

As an illustration: The curve corresponding to the function which is the square root of x may be drawn on a sheet of paper by a very simple and mechanical contrivance. Supposing it so drawn, then we can ascertain the square roots of numbers by merely measuring lines on a diagram.

The form of the curve in question is shown in Fig. (3). For each positive value of x there are two perpendiculars, $P\,N$ $P'N$, equal in length but on opposite sides of $O\,X$, corresponding to the fact that every positive number has two square roots equal in magnitude but opposite in sign.

§ 6. It will be useful to regard the lines ON and PN, which are called respectively the abscissa and ordinate of the point P, from a different point of view. Starting with a given line of reference XOX', the position of any point on the diagram is determined when its abscissa and ordinate are given. The abscissa we have denoted by the letter x, the ordinate is usually denoted by the letter y. Thus the point whose abscissa is 3 and ordinate 10—in other words, for which $x=3$, $y=10$—is the point P, found by measuring along OX a line $ON=3$ and drawing through N a line NP perpendicular to ON and equal to 10. This point might, however, equally well have been found as follows:—Take another line of reference YOY' drawn through O at right angles to XOX', measure along OY a line $OM=10$ and through M draw MP perpendicular to OY and equal to 3. We see, in fact, that the abscissa (denoted by x) is the distance of the point P from the line YOY', called the axis of y; and the ordinate (denoted by y) is the distance of the point P from the line XOX', called the axis of x.





Looked at in this way there is no distinction in theory between the abscissa and ordinate of a point. The two are called the co-ordinates of the point.

As an example, consider the function which is the square of x. For this function we have the equation

$$y=x^2$$

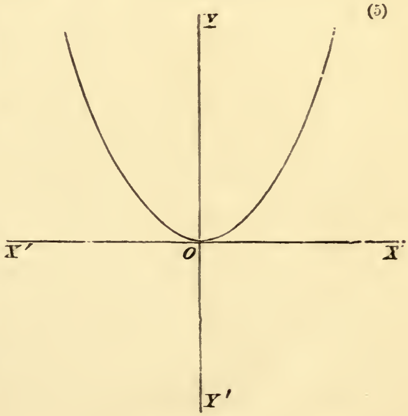
which may be called the equation to the curve which represents the function under consideration, since it is the equation which connects the co-ordinates of every point on this curve. Giving to the independent variable x any values we please, and determining from the equation the corresponding values of y, we may obtain any number of points on the curve. We may, however, proceed differently. The equation may be written

$$x=\pm\sqrt{y},$$

and we may now take y for the independent variable, assign to it any values we please, and determine the corresponding values of x. Of course we shall merely obtain the same curve over again.

It is easy to see from either form of the equation that the curve lies entirely above the axis of x and is symmetrical about the axis of y. For from the equation $y=x^2$ we see that y is essentially positive, and that two values of x, which differ only in sign, give the same value of y. Again, from $x=\pm\sqrt{y}$ we see that y, in order to have a real square root must be positive, and that for every positive value of y there are two of x, differing only in sign.

The curve is that given in Fig. (5). If the scale of measurement is the same for x and y, and the same scale is employed in Fig. (3), the curve in Fig. (5) is the same as the curve in Fig. (3) turned through a right angle about the point O. In fact, the equations of the two curves being $y^2=x$ and $y=x^2$, one is situated with regard to the axes of x and y in the same way that the other is situated with regard to the axes of y and x.



CHAPTER I.

§ 1. In arithmetic the numerical difference between two quantities is the amount by which the greater exceeds the less. Thus the difference between 5 and 7 is 2 and the greater number may be obtained by adding the difference to the smaller. In algebra the amount by which b differs from a is defined to be $b-a$ and this quantity may of course be either positive or negative. When we speak of the difference between any two quantities a and b, we shall mean the quantity $b-a$, or the quantity which must be added to the former to get the latter.

§ 2. Suppose we have a series of quantities proceeding according to some given law, *e.g.*, the fourth powers of the natural numbers. By subtraction we may find the difference between any one and the next following; *e.g.*, the difference between 3^4 and 4^4 is $256-81=175$, and 4^4 may be obtained by adding 175 to 3^4. These quantities and their differences may be arranged in a table thus:—

x	x^4	Δ	Δ^2	Δ^3	Δ^4
1	1				
		15			
2	16		50		
		65		60	
3	81		110		24
		175		84	
4	256		194		24
		369		108	
5	625		302		24
		671		132	
6	1,296		434		
		1,105			
7	2,401				

Here the first column gives the natural numbers, the second their fourth powers, and the third the differences of the fourth powers. In the same way that the differences of the fourth powers are found, may also be found the differences of these differences. These last are written in the fourth column and are called the second differences of the quantities in the second column. In the same way may be found for this or any other series of quantities the differences of the second differences called the third differences, and so on, the process of differencing being capable of repetition without limit. It may of course happen that after a certain point all the differences vanish. In the illustration given above it will be found that the fourth differences are all equal, and therefore the fifth and higher differences are zero.* It is obvious that the mth differences of the nth differences of a series of quantities are the $\overline{m+n}$th differences of the series of quantities.

§ 3. The attention of the student is drawn to the following table or scheme of differences :—

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-5h & u_{-5h} & & & & \\
 & & \Delta u_{-5h} & & & \\
-4h & u_{-4h} & & \Delta^2 u_{-5h} & & \\
 & & \Delta u_{-4h} & & \Delta^3 u_{-5h} & \\
-3h & u_{-3h} & & \Delta^2 u_{-4h} & & \Delta^4 u_{-5h} \ldots \\
 & & \Delta u_{-3h} & & \Delta^3 u_{-4h} & \\
-2h & u_{-2h} & & \Delta^2 u_{-3h} & & \Delta^4 u_{-4h} \ldots \\
 & & \Delta u_{-2h} & & \Delta^3 u_{-3h} & \\
-\ h & u_{-h} & & \Delta^2 u_{-2h} & & \Delta^4 u_{-3h} \ldots \\
 & & \Delta u_{-h} & & \Delta^3 u_{-2h} & \\
0 & u_0 & & \Delta^2 u_{-h} & & \Delta^4 u_{-2h} \ldots \\
 & & \Delta u_0 & & \Delta^3 u_{-h} & \\
h & u_h & & \Delta^2 u_0 & & \Delta^4 u_{-h} \ldots \\
 & & \Delta u_h & & \Delta^3 u_0 & \\
2h & u_{2h} & & \Delta^2 u_h & & \Delta^4 u_0 \ldots \\
 & & \Delta u_{2h} & & \Delta^3 u_h & \vdots \\
3h & u_{3h} & & \Delta^2 u_{2h} & \vdots & \\
 & & \Delta u_{3h} & \vdots & & \\
4h & u_{4h} & \vdots & & & \\
\vdots & \vdots & & & &
\end{array}
$$

In the first column are a series of equidistant values of x, in the second the corresponding values of a function of x denoted by u_x, in the third the first differences of the quantities in the second column, which may be found by subtraction thus:—

$$\Delta u_{-5h} = u_{-4h} - u_{-5h} \ldots \Delta u_{-4h} = u_{-3h} - u_{-4h} \ldots \Delta u_{3h} = u_{4h} - u_{3h} \ldots$$

In the fourth column are given the differences of the quantities of the third column, which are called the second differences of the quantities in the second column, and so on.

§ 4. If we are given the first of a series of values of a function and all the first differences, we can, by addition, form the terms of the series. Let for instance $u_0, u_h, u_{2h} \ldots$ denote the series of values of the function, and suppose we have given

$$u_0, \Delta u_0, \Delta u_h, \Delta u_{2h} \ldots$$

Adding Δu_0 to u_0, we get u_h; then adding Δu_h to u_h, we get u_{2h}, and so on.

* We do not say that there are no fifth or higher differences, but that the fifth and higher differences vanish or are all zero. This may appear a distinction without a difference, but it will enable the student to better appreciate some of the formulæ which occur later on.

Or, if we have given $u_0, \Delta u_0, \Delta^2 u_0 \ldots \Delta^n u_0$, and all the $\overline{n+1}$th differences, we can form the series $u_0, u_h, u_{2h} \ldots$ by addition. For by adding in succession the $\overline{n+1}$th differences to $\Delta^n u_0$ we form the table of nth differences, then adding these in succession to $\Delta^{n-1} u_0$ we form the $\overline{n-1}$th differences, and so on.

Ex.: Given $u_0=1$, $\Delta u_0=15$, $\Delta^2 u_0=50$, $\Delta^3 u_0=60$, $\Delta^4 u_n=24$ for all values of n, find $u_0, u_h, u_{2h} \ldots$

Here by repeated addition of 24 to 60 we get the column of third differences 60, 84, 108 . . . ; then adding these in succession to 50 we obtain the column of second differences 110, 194, 302 . . . , and so on, finally obtaining for $u_h, u_{2h}, u_{3h} \ldots$ the values 16, 81, 256 . . . . The figures are shown in the table paragraph (2).

With respect to the series of quantities $u_0, u_h, u_{2h}, u_{3h} \ldots$, $u_0, \Delta u_0, \Delta^2 u_0 \ldots$ are called the initial term and initial differences, or the leading term and leading differences. The leading term and leading differences for the quantities $u_{nh}, u_{\overline{n+1}.h}, u_{\overline{n+2}.h} \ldots$ are of course $u_{nh}, \Delta u_{nh}, \Delta^2 u_{nh} \ldots$.

§ 5. In sections (1) to (4) we have regarded the differences of any function as found by subtraction from a series of equidistant values of it supposed given. It will be useful to consider them from a somewhat different point of view.

Let us take any function of x denoted by u_x and find the increment of the function corresponding to an increase h in the variable x, in other words, find the difference between u_x and u_{x+h}. This is obviously $u_{x+h}-u_x$. The expression $u_{x+h}-u_x$ is called the first difference of u_x corresponding to the increment h of the independent variable x and is denoted by Δu_x. In the same way Δu_x being a function of x we may find its difference corresponding to the increment h of x. This is called the second difference of the original function and is denoted by $\Delta^2 u_x$. Similarly, the third, fourth, &c. differences may be found, denoted by $\Delta^3 u_x$, $\Delta^4 u_x \ldots$.

Ex.: Take the function x^4; then

$$\begin{aligned}
\Delta x^4 &= (x+h)^4 - x^4 \\
&= 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \\
\Delta^2 x^4 &= 4(x+h)^3h + 6(x+h)^2h^2 + 4(x+h)h^3 + h^4 \\
&\quad -4x^3h - 6x^2h^2 - 4xh^3 - h^4 \\
&= 12x^2h^2 + 24xh^3 + 14h^4.
\end{aligned}$$

Similarly
$$\Delta^3 x^4 = 24xh^3 + 36h^4$$
$$\Delta^4 x^4 = 24h^4$$
$$\Delta^5 x^4 = 0.$$

If h is taken $=1$, we have

$$\delta x^4 = 4x^3 + 6x^2 + 4x + 1$$
$$\delta^2 x^4 = 12x^2 + 24x + 14$$
$$\delta^3 x^4 = 24x + 36$$
$$\delta^4 x^4 = 24.$$

from which, putting $x = 1, 2, 3 \ldots$ we form the table of differences given, p. ara. (iv) .

It is to be noted that in forming the successive differences of any function, the increment of x, which might also be called the difference of x, denoted above by h, is taken to be the same throughout. In practice it usually has the value unity, but as the investigation of the more elementary theorems of the subject is not simplified by assuming $h=1$, we shall, unless it is otherwise stated, suppose its value unrestricted.

§ 6. The following illustrations of the subject are worthy of attention. The demonstrations follow directly from the definition of a difference, and they should be tried as exercises before reading the proofs.

(1). If a is a constant quantity, that is a quantity independent of x,

$$\Delta a u_x = a \Delta u_x.$$

(2). If u_x and v_x are two functions of x,

(a) $\Delta(u_x \pm v_x) = \Delta u_x \pm \Delta v_x$

(β) $$\Delta u_x v_x = u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \Delta v_x$$

(γ) $$\Delta \frac{u_x}{v_x} = \frac{v_x \Delta u_x - u_x \Delta v_x}{v_x v_{x+h}}.$$

(3). $$\Delta \log x = \log\left(1 + \frac{h}{x}\right)$$

(4). $$\Delta^n a^x = (a^h - 1)^n a^x$$

(5). $$\Delta^n x^n = \underline{|n} h^n.$$

(1). $$\Delta a u_x = a u_{x+h} - a u_x$$
$$= a(u_{x+h} - u_x)$$
$$= a\Delta u_x$$

(2). (a) $$\Delta(u_x \pm v_x) = u_{x+h} \pm v_{x+h} - (u_x \pm v_x)$$
$$= u_{x+h} - u_x \pm (v_{x+h} - v_x)$$
$$= \Delta u_x \pm \Delta v_x$$

(β) $$\Delta u_x v_x = u_{x+h} v_{x+h} - u_x v_x$$
$$= (u_x + \Delta u_x)(v_x + \Delta v_x) - u_x v_x$$
$$= u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \Delta v_x$$

(γ) $$\Delta \frac{u_x}{v_x} = \frac{u_{x+h}}{v_{x+h}} - \frac{u_x}{v_x}$$
$$= \frac{v_x(u_x + \Delta u_x) - u_x(v_x + \Delta v_x)}{v_x v_{x+h}}$$
$$= \frac{v_x \Delta u_x - u_x \Delta v_x}{v_x v_{x+h}}$$

(3). $$\Delta \log x = \log(x+h) - \log x$$
$$= \log \frac{x+h}{x}$$
$$= \log\left(1 + \frac{h}{x}\right)$$

(4). $$\Delta a^x = a^{x+h} - a^x$$
$$= (a^h - 1)a^x$$
$$\Delta^2 a^x = \Delta(a^h - 1)a^x$$
$$= (a^h - 1)\Delta a^x$$
$$= (a^h - 1)(a^h - 1)a^x$$
$$= (a^h - 1)^2 a^x$$

Similarly $$\Delta^3 a^x = (a^h - 1)^2 \Delta a^x$$
$$= (a^h - 1)^3 a_x$$

and so on.

(5). This is a particular case of the following more general theorem.

Let $u_x = \mathrm{A}x^n + \mathrm{B}x^{n-1} + \mathrm{C}x^{n-2} + \ldots$, a rational integral algebraic function of x of degree n. Then, if $r < n$, $\Delta^r x^n$ is a rational integral algebraic function of x of degree $n-r$ in which the term involving x^{n-r} is $An\,.\,\overline{n-1}\,.\,\overline{n-2}\,\ldots\,\overline{n-r+1}x^{n-r}h^r$; if $r=n$, $\Delta^r x^n = A\lfloor n h^n$; if $r>n$, $\Delta^r x^n = 0$. The proof of this theorem is as follows:—

$$\begin{aligned}\Delta u_x &= A(x+h)^n + B(x+h)^{n-1} + \ldots\\ &\qquad - Ax^n - Bx^{n-1} - \ldots\\ &= Anx^{n-1}h + B_1 x^{n-2} + C_1 x^{n-3} + \ldots\end{aligned}$$

Differencing again we obtain in the same way,

$$\Delta^2 u_x = An\,.\,\overline{n-1}x^{n-2}h^2 + B_2 x^{n-3} + C_2 x^{n-4} + \ldots$$

similarly,

$$\Delta^3 u_x = An\,.\,\overline{n-1}\,.\,\overline{n-2}x^{n-3}h^3 + B_3 x^{n-4} + C_3 x^{n-5} + \ldots$$

and so on. Thus if $r<n$ we obtain on differencing r times

$$\Delta^r u_x = An\,.\,\overline{n-1}\,\ldots\,\overline{n-r+1}x^{n-r}h^r + B_r x^{n-r-1} + \ldots.$$

Again, differencing n times we get

$$\Delta^n u_x = An\,.\,\overline{n-1}\,.\,\overline{n-2}\,\ldots\,1\,.\,h^n;$$

and this quantity being independent of x, all higher differences vanish.

§ 7. To express $\Delta^n u_x$ in terms of u_x and its successive values u_{x+h}, u_{x+2h} ....

$$\begin{aligned}\Delta u_x &= u_{x+h} - u_x\\ \Delta^2 u_x &= u_{x+2h} - u_{x+h} - (u_{x+h} - u_x)\\ &= u_{x+2h} - 2u_{x+h} + u_x\\ \Delta^3 u_x &= u_{x+3h} - 2u_{x+2h} + u_{x+h}\\ &\qquad - (u_{x+2h} - 2u_{x+h} + u_x)\\ &= u_{x+3h} - 3u_{x+2h} + 3u_{x+h} - u_x.\end{aligned}$$

From the above examples it might be inferred that we should have generally

$$\Delta^n u_x = u_{x+nh} - p_1 u_{x+\overline{n-1}h} + p_2 u_{x+\overline{n-2}.h} - \ldots$$

where p_1, p_2, ... are the numerical values of the coefficients in the expansion of $(a-x)^n$. Let us assume, then, that this law holds for n, and examine whether it will hold for $n+1$. Differencing we obtain

$$\begin{aligned}\Delta\Delta^n u_x &= u_{x+\overline{n+1}h} - p_1 u_{x+nh} + p_2 u_{x+\overline{n-1}h} - \ldots\\ &\qquad - \{u_{x+nh} - p_1 u_{x+\overline{n-1}h} + p_2 u_{x+\overline{n-2}h} - \ldots\}\end{aligned}$$

i.e.,

$$\Delta^{n+1}u_x = u_{x+\overline{n+1}h} - (1+p_1)u_{x+nh} + (p_1+p_2)u_{x+\overline{n-1}h} - (p_2+p_3)u_{x+\overline{n-2}h} + \ldots$$

Now, if we multiply both sides of the equation

$$(a-x)^n = a^n - p_1a^{n-1}x + p_2a^{n-2}x^2 - p_3a^{n-3}x^3 + \ldots$$

by $a-x$, we obtain

$$(a-x)^{n+1} = a^{n+1} - (1+p_1)a^nx + (p_1+p_2)a^{n-1}x^2 - (p_2+p_3)a^{n-2}x^3 + \ldots$$

Thus the coefficients in the series for $\Delta^{n+1}u_x$ are the same as those in the series for $(a-x)^{n+1}$. If, then, the law holds for n it holds for $n+1$. But we have seen that the law holds for Δu_x, therefore it holds for Δ^2u_x, and therefore for Δ^3u_x, and so on universally. That is, we have

$$\Delta^n u_x = u_{x+nh} - nu_{x+\overline{n-1}h} + \frac{n.\overline{n-1}}{1.2}u_{x+\overline{n-2}h} - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}u_{x+\overline{n-3}h} + \ldots \quad (1)$$

§ 8. To express u_{x+nh} in terms of u_x and its first n leading differences, *i.e.*, in terms of u_x, Δu_x, $\Delta^2 u_x \ldots \Delta^n u_x$.

$$u_{x+h} = u_x + \Delta u_x$$
$$u_{x+2h} = u_{x+h} + \Delta u_{x+h}$$
$$= u_x + \Delta u_x + \Delta(u_x + \Delta u_x)$$
$$= u_x + 2\Delta u_x + \Delta^2 u_x.$$

Similarly $\qquad u_{x+3h} = u_x + 3\Delta u_x + 3\Delta^2 u_x + \Delta^3 u_x,$

and in the same way as in the preceding proposition it may be shown that

$$u_{x+nh} = u_x + n\Delta u_x + \frac{n.\overline{n-1}}{1.2}\Delta^2 u_x + \ldots + n\Delta^{n-1}u_x + \Delta^n u_x \quad . \quad . \quad (2)$$

§ 9. In § 2 we have supposed a series of quantities *proceeding according to some given law.* We may, however, take a series of numbers at random and obtain from them their first, second . . . differences by subtraction. These quantities and their differences may still be symbolised as in the table of § 3, the suffix nh of any one of the quantities u_{nh} merely denoting its position in the series. In sections 4, 5, 7 and 8, no assumption was made as to the nature of the law of formation of the quantities considered, and they therefore apply to a series of numbers chosen at random and their differences. Of such a series of numbers it is perhaps preferable to say, not that they are numbers chosen at random, but that the law of their formation is that of arbitrary selection.

CHAPTER II.

Interpolation.

§ 1. It frequently arises, in the calculation of tables and in physical and statistical researches, that certain values of a function are found by experiment, observation, or calculation, and it is required to obtain from them approximately other values of the function. For example, we might be given as the result of observation the number of persons attaining the ages 15, 20, 25 . . . out of 100,000 alive at age 10, and require to find approximately the numbers attaining the intermediate ages 11, 12, 13, 14, 16, 17 . . . .

To denote this process of approximating to unknown values of a function or terms of a series by means of other values which are given we make use of the expression *interpolation*.

When we do not know the general form of a function u_x—*i.e.*, either the algebraic expression for it or the curve which represents it—but are given merely a series of values of it corresponding to a series of different values of x—that is, are given merely a series of isolated points on the curve which represents it—the problem of finding the values of the function for other values of x or of determining other points on the curve is clearly indeterminate, since between any two consecutive given points we may draw any number of lines we please, and thus we may get an infinite number of curves passing through the given series of points. It is therefore necessary to make some assumption in order to have a problem with a definite solution. This assumption is usually made as follows:—

§ 2. In the case of most of the tabulated functions—*e.g.*, logarithms—the successive orders of differences rapidly diminish, and this will usually be found to be the case with the functions with which the actuary and the statistician are concerned.

When m particular values of a function are given, and it is required to find other values or a general expression for the function, it is usual to assume that the function can be completely represented by a rational integral algebraic expression in x of degree

$m-1$. This, as we have seen (Ex. 5, p. 15), is equivalent to assuming that the mth and all higher differences vanish. Proceeding on this hypothesis we shall first consider and illustrate two particular problems in interpolation, and afterwards (Chapter IV.) give a few more general theorems on the subject.

§ 3. I.—Given m equidistant values $u_0, u_h, u_{2h} \ldots u_{\overline{m-1}h}$ of a function, to find a general expression for it—that is, to find its value for any other value of the independent variable.

By the equation (2), p. 17, we have (which see)

$$u_{nh} = u_0 + n\Delta u_0 + \frac{n.\overline{n-1}}{1.2}\Delta^2 u_0 + \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}\Delta^3 u_0 + \ldots$$

or, using the symbol x to denote nh,

$$u_x = u_0 + \frac{x}{h}\Delta u_0 + \frac{x.\overline{x-h}}{2h^2}\Delta^2 u_0 + \ldots \quad . \quad . \quad . \quad (3)$$

where it must be remembered that the only restriction on x is that it must be a multiple of h. The last term of the series is

$$\frac{x.\overline{x-h} \ldots (x-\overline{m-2}h)}{\underline{|m-1}\,h^{m-1}}\Delta^{m-1}u_0$$

since under our hypothesis $\Delta^m u_0$ and all the higher differences vanish.

Equation (3) gives us a general expression for u_x when $\frac{x}{h}$ is any positive integer. But the form of the expression for u_x is, of course, independent of this restriction as to the value of x. Having found the general expression for u_x when x has any one of the values $0, h, 2h, 3h \ldots$ which are infinite in number, we infer that this is the expression for u_x when the value of x is unrestricted. We therefore have whatever may be the value of x,

$$u_x = u_0 + \frac{x}{h}\Delta u_0 + \frac{x.(x-h)}{1.2h^2}\Delta^2 u_0 + \ldots$$
$$+ \frac{x.(x-h) \ldots (x-\overline{m-2}h)}{\underline{|m-1}\,h^{m-1}}\Delta^{m-1}u_0 \quad . \quad . \quad . \quad . \quad (3)$$

It will be observed that the above expression is of degree $m-1$ in x, and giving to x in succession the values $0, h, 2h \ldots \overline{m-1}h$, it takes the values $u_0, u_h, u_{2h} \ldots u_{\overline{m-1}h}$.

The student should bear in mind that this expression for u_x is exact only when u_x is known to be an expression of the $\overline{m-1}$th degree in x. If, as will generally be the case, we know nothing of the function except the m given values of it, we can only say that the series represents it approximately, since $\Delta^m u_0$, $\Delta^{m+1}u_0$ . . . have all been neglected.

In numerical applications it will usually be convenient to write the equation in the form

$$u_x = u_0 + \frac{x}{h}\Delta u_0 + \frac{\frac{x}{h}\left(\frac{x}{h}-1\right)}{\underline{|2}}\Delta^2 u_0 + \ldots + \frac{\frac{x}{h}\left(\frac{x}{h}-1\right)\ldots\left(\frac{x}{h}-m+2\right)}{\underline{|m-1}}\Delta^{m-1}u_0 \quad . \; . \; (3a$$

$u_0 = l_{10}$
$u_{10} = l_{20}$
$= u_0 + \frac{1}{10}\Delta u_0$

We may regard x as the distance of u_x from the first of the given values of the function, and h as the distance between any two consecutive given values.

§ 4. *Ex.*: The Northampton 3 per-cent annuities for ages 21, 25, 29, 33, and 37, are respectively 18·4708, 17·8144, 17·1070, 16·3432, and 15·5154. Find the annuity for age 30.

Taking 15·5154 as the initial value of the function, we have $h=4$, $x=7$, and $\frac{x}{h}=1\cdot75$. By subtraction we obtain the following table of differences:—

			Δ	Δ^2	Δ^3	Δ^4
a_{37} =	u_0	15·5154	·8278	−·0640	·0076	−·0022
a_{33} =	u_4	16·3432	·7638	−·0564	·0054	
a_{29} =	u_8	17·1070	·7074	−·0510		
a_{25} =	u_{12}	17·8144	·6564			
a_{21} =	u_{16}	18·4708				

Whence $u_0=15\cdot5154$, $\Delta u_0=\cdot8278$, $\Delta^2 u_0=-\cdot064$, $\Delta^3 u_0=\cdot0076$, $\Delta^4 u_0=-\cdot0022$, and the expression for u_x takes the value

$u_0 + \frac{7}{4}\Delta u_0 - \ldots$

$$u_7 = 15\cdot5154 + 1\cdot75\times\cdot8278 - \frac{1\cdot75\times\cdot75}{2}\times\cdot064$$

$$-\frac{1\cdot75\times\cdot75\times\cdot25}{6}\times\cdot0076 - \frac{1\cdot75\times\cdot75\times\cdot25\times1\cdot25}{24}\times\cdot0022$$

$$=16\cdot9216.$$

(The value given in *Jones on Annuities*, vol. i, p. 244, is 16·9217.)

In the above solution the initial value of u_x was taken to be that corresponding to age 37 and increments of x to decrements of age. The problem might equally well have been solved, starting from age 21 as the initial point and taking increments of x to correspond to increments of age. If this had been done, the signs of the odd differences would have been reversed. We should have had $x=9$, $u_0=18{\cdot}4708$, $\Delta u_0=-{\cdot}6564$, $\Delta^2 u_0=-{\cdot}051$, $\Delta^3 u_0=-{\cdot}0054$, and $\Delta^4 u_0=-{\cdot}0022$.

§ 5. If we choose to denote the initial value of u by u_k, the equation (3) becomes

$$u_{k+x}=u_k+\frac{x}{h}\Delta u_k+\frac{x.\overline{x-h}}{\underline{|2}h^2}\Delta^2 u_k+\ldots$$

This is quite obvious when we bear in mind the remark made at the end of § 3 as to the meaning of x.

§ 6. II.—Suppose the given values of the function do not form a complete series of equidistant values in consequence of the absence of one or more terms. Let it be required to supply these missing terms.

Let u_0, u_h, u_{2h} . . . denote the complete series, of which m terms are supposed given. If one term only is wanting, the number of terms in the complete series is $m+1$, and, to find the missing term we put $\Delta^m u_0=0$, or, by equation (1)

$$u_{mh}-mu_{\overline{m-1}h}+\frac{m.\overline{m-1}}{1.2}u_{\overline{m-2}h}-\ \ldots\ +(-1)^m u_0=0.$$

In this equation all but one of the $m+1$ quantities $u_0, u_h, u_{2h}\ldots u_{mh}$ are known, and it is therefore a simple equation to find the one required.

If two terms are deficient the number of terms in the complete series being $m+2$ we must also put $\Delta^m u_h=0$, and thus get, to find the two unknown quantities, the two equations

$$u_{mh}-mu_{\overline{m-1}h}+\frac{m.\overline{m-1}}{1.2}u_{\overline{m-2}h}-\ \ldots\ +(-1)^m u_0=0,$$

$$u_{\overline{m+1}h}-mu_{mh}+\frac{m.\overline{m-1}}{1.2}u_{\overline{m-1}h}-\ \ldots\ +(-1)^m u_h=0;$$

and generally, if r terms are deficient, the total number of terms being $m+r$, the r equations to find them are got by equating separately to zero the series for $\Delta^m u_0$, $\Delta^m u_h$, $\Delta^m u_{2h}$ . . ., $\Delta^m u_{\overline{r-1}h}$.

Ex.: Given u_0, u_2, u_3, u_5, u_6, find u_1 and u_4.

Putting $\Delta^5 u_0$ and $\Delta^5 u_1=0$, we have

$$u_5-5u_4+10u_3-10u_2+5u_1-u_0=0$$
$$u_6-5u_5+10u_4-10u_3+5u_2-u_1=0$$

From which we obtain

$$u_4=\tfrac{1}{45}\{u_0-15u_2+40u_3+24u_5-5u_6\}$$
$$u_1=\tfrac{1}{9}\{2u_0+15u_2-10u_3+3u_5-u_6\}$$

§ 7. The method given in the preceding section will, of course, always succeed. In special cases, however, other processes of calculation may be employed with advantage. For these no general rule can be given. The following is an example (*vide Journal of the Institute of Actuaries*, vol. xv, pp. 394, 395). If $u_0 = 100{,}000$, $u_7 = 97{,}189$, $u_8 = 96{,}720$, $u_9 = 96{,}195$, find u_1, $u_2 \ldots u_6$.

Here we assume that 4th differences vanish, and therefore 3rd differences are constant. Reverse the series and let the terms in reverse order be denoted by $v_0, v_1, v_2 \ldots v_9$. By subtraction we form the following difference table:—

Δ Δ^2

v_0	96,195		
		525	
v_1	96,720		-56
		469	
v_2	97,189		

But by equation (2) or (3)

$$v_n = v_0 + n\Delta v_0 + \frac{n.\overline{n-1}}{1.2}\Delta^2 v_0 + \frac{n.\overline{n-1}.\overline{n-2}}{\underline{|3}}\Delta^3 v_0$$

whence, putting $n=9$, we have

$$100{,}000 = 96{,}195 + 9 \times 525 - 36 \times 56 + 84 \times \Delta^3 v_0,$$

which gives $\Delta^3 v_0 = 13{\cdot}05.$

Having now obtained the leading quantities $v_0 = 96{,}195$, $\Delta v_0 = 525$, $\Delta^2 v_0 = -56$, $\Delta^3 v_x = 13{\cdot}05$, the table given below is formed by addition.

		Δ	Δ^2	Δ^3
$u_9 = v_0$	96,195			
		525·00		
$u_8 = v_1$	96,720		$-56{\cdot}00$	
		469·00		13·05
$u_7 = v_2$	97,189		$-42{\cdot}95$	
		426·05		
$u_6 = v_3$	97,615		$-29{\cdot}90$	
		396·15		
$u_5 = v_4$	98,011		$-16{\cdot}85$	
		379·30		
$u_4 = v_5$	98,391		$-3{\cdot}80$	
		375·50		
$u_3 = v_6$	98,766		9·25	
		384·75		
$u_2 = v_7$	99,151		22·30	
		407·05		
$u_1 = v_8$	99,558		35·35	
		442·40		
$u_0 = v_9$	100,000			

CHAPTER III.

Illustrations of the Subject.

§ 1. To sum the series formed by n consecutive equidistant values of a given function of x. Let u_x denote the function, and $u_r, u_{r+h} \ldots u_{r+\overline{n-1}h}$ the series of terms to be summed. Let v_x be such a function of x that $\Delta v_x = u_x$. Then

$$\begin{aligned} &u_r + u_{r+h} + \ldots + u_{r+n-1h} \\ &= \Delta v_r + \Delta v_{r+h} + \ldots + \Delta v_{r+\overline{n-1}h} \\ &= v_{r+nh} - v_r . \end{aligned}$$

So that the sum of the series can be found if the expression for v_x can be found.

Ex.: Find the sum of

$$\begin{aligned} &(ar+b)(a\overline{r+1}+b) \ldots (a\overline{r+m-1}+b) \\ &+ (a\overline{r+1}+b)(a\overline{r+2}+b) \ldots (a\overline{r+m}+b) \\ &+ \ldots + (a.\overline{r+n-1}+b)(a\overline{r+n}+b) \ldots (a.\overline{r+m+n-2}+b). \end{aligned}$$

Denoting the function $(ax+b)(a\overline{x+1}+b) \ldots (a\overline{x+m-1}+b)$ in which the number of factors is m by $u_x^{(m)}$, and taking the increment of x unity we have

$$\begin{aligned} \delta u_x^{(m)} &= (a\overline{x+1}+b)(\ ) \ldots (a\overline{x+m}+b) - (ax+b)(\ ) \ldots (a\overline{x+m-1}+b) \\ &= (a.\overline{x+1}+b)(a.\overline{x+2}+b) \ldots (a\overline{x+m-1}+b).am \\ &= amu_{x+1}^{(m-1)}. \end{aligned}$$

Therefore, changing m into $m+1$ and x into $x-1$, we have $\frac{1}{a(m+1)} \delta u_{x-1}^{(m+1)} = u_x^{(m)}$; that is, $\frac{1}{a(m+1)} u_{x-1}^{(m+1)}$ is a function whose difference is $u_x^{(m)}$. The sum of the series is therefore

$$\begin{aligned} &\frac{1}{a(m+1)} \left\{ u_{r+n-1}^{(m+1)} - u_{r-1}^{(m+1)} \right\} \\ &= \frac{1}{a(m+1)} \left\{ (a\overline{r+n-1}+b)(\ ) \ldots (a.\overline{r+n+m-1}+b) \right. \\ &\qquad \left. -(a.\overline{r-1}+b)(\ ) \ldots (a.\overline{r+m-1}+b) \right\}. \end{aligned}$$

§ 2. To find the sum of a series of equidistant values of any rational integral algebraic function of x.

Let the terms of the series to be summed be denoted by $u_0, u_1, u_2, \ldots u_{n-1}$, and suppose the increment of x in passing from any one to the next is h. Let $S_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$, S_0 being equal to zero. Then, Δ having reference to the increment h in x or 1 in n,

$$\Delta S_n = u_n ;$$

Therefore
$$\Delta^2 S_n = \Delta u_n$$
$$\Delta^3 S_n = \Delta^2 u_n, \text{ \&c.}$$

Now, by equation (2), Chap. I,

$$S_n = S_0 + n\Delta S_0 + \frac{n.\overline{n-1}}{1\cdot 2}\Delta^2 S_0 + \ldots$$

$$= nu_0 + \frac{n.\overline{n-1}}{1\cdot 2}\Delta u_0 + \frac{n.\overline{n-1}.\overline{n-2}}{|\!\underline{3}}\Delta^2 u_0 + \ldots .$$

Substituting in this equation their values for $u_0, \Delta u_0, \Delta^2 u_0 \ldots$, we obtain S_n or the required sum.

Ex.: Find the sum of the series

$$K^3 + (K+h)^3 + (K+2h)^3 + \ldots + \{K + \overline{n-1}h\}^3$$

By the formula just found the sum is equal to

$$nK^3 + \frac{n.\overline{n-1}}{|\!\underline{2}}(3K^2h + 3Kh^2 + h^3) + \frac{n.\overline{n-1}.\overline{n-2}}{|\!\underline{3}} 6Kh^2 + 6h^3)$$
$$+ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{|\!\underline{4}} 6h^3$$

As a particular case, putting $K = h = 1$, we have

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n.\overline{n+1}}{2}\right\}^2 .$$

§ 3. The differences of the various positive integral powers of x, in which the increment of x is taken to be unity and x is put equal to zero after evaluation, are called the Differences of Nothing. The general symbol for them is $\delta^m 0^n$. To obtain $\delta^m 0^n$, x^n is differenced m times, and in the result x is put $= 0$.

These numbers are of importance in the study of Finite Differences, and we shall see a use for them later on. We proceed to investigate a formula by which they can be easily calculated.

By equation (1), Chap. I, putting $u_x = x^n$, we have

$$\delta^m x^n = (x+m)^n - m(x+m-1)^n + \frac{m.\overline{m-1}}{\underline{|2}}(x+m-2)^n - \ldots \quad (a).$$

$$\therefore\ \delta^m 0^n = m^n - m(m-1)^n + \frac{m.\overline{m-1}}{\underline{|2}}(m-2)^n - \ \ldots$$

$$= m\left\{m^{n-1} - \overline{m-1}(m-1)^{n-1} + \frac{\overline{m-1}.\overline{m-2}}{\underline{|2}}(m-2)^{n-1} - \ldots\right\}$$

$$= m\delta^{m-1}1^{n-1}, \text{ (by equation } (a)).$$

But since we have always $u_{x+h} = u_x + \Delta u_x$, therefore

$$\delta^{m-1}1^{n-1} = \delta^{m-1}0^{n-1} + \delta\delta^{m-1}0^{n-1};$$

therefore $$\delta^m 0^n = m\{\delta^{m-1}0^{n-1} + \delta^m 0^{n-1}\}.$$

By means of this equation the Differences of Nothing may be tabulated without difficulty, as in the following diagram:—

	0^1	0^2	0^3	0^4	0^5	0^6
δ_1	1	1	1	1	1	1
δ_2	0	2	6	14	30	62
δ_3	0	0	6	36	150	540
δ_4	0	0	0	24	240	1,560
δ_5	0	0	0	0	120	1,800
δ_6	0	0	0	0	0	720

Each first difference is unity, since $1^n - 0^n = 1$. Again, if the diagonal of the diagram passing through the right bottom corner be drawn (indicated by the dotted line), all the terms below it are zero, since $\delta^m x^n = 0$ if $m > n$. The rows may be completed in succession, working from left to right as follows: To find a term in the mth row, take the term to the left of it and the term above that, add the two together, and multiply the sum by m. Thus

$$\delta^2 0^2 = 2(0+1) = \ 2$$

$$\delta^2 0^3 = 2(2+1) = \ 6$$

$$\delta^2 0^4 = 2(6+1) = 14 \text{ \&c.}$$

Similarly $$\delta^3 0^3 = 3(0+2) = \ 6$$

$$\delta^3 0^4 = 3(6+6) = 36 \text{ \&c.}$$

CHAPTER IV.

Interpolation Formulæ.

§ 1. Given a series of equidistant values of a function, to insert between each two consecutive values $n-1$ equidistant terms so as to divide each interval into n equal intervals.

Denoting any pair of consecutive given values by u_k and u_{k+n}, the terms to be interpolated between them will be denoted by $u_{k+1}, u_{k+2} \ldots u_{k+n-1}$. The solution of the problem is obtained from equation (3). Taking the form of it given in § 5 of Chapter II we have

$$u_{k+x} = u_k + \frac{x}{n}\Delta u_k + \frac{x.\overline{x-n}}{\lfloor 2 \, n^2}\Delta^2 u_k + \ldots .$$

In this equation putting x in succession equal to $1, 2 \ldots \overline{n-1}$, we obtain the values of the terms to be interpolated in the interval between u_k and u_{k+n}.

To calculate each term separately by a direct application of this formula would be a tedious operation. The process may be shortened as follows.

Let $\delta, \delta^2, \delta^3 \ldots$ denote first, second ... differences when the increment of x is unity. If $\delta u_k, \delta^2 u_k \ldots$ are found the terms to be interpolated can be formed by addition.

To find $\delta^r u_k$ we must take the rth difference of u_{k+x}, that is of the series $u_k + \frac{x}{n}\Delta u_k + \frac{x.\overline{x-n}}{\lfloor 2 \, n^2}\Delta^2 u_k + \ldots$, with respect to x, and in the result put $x=0$. The only quantities which have to be differenced are the various powers of x which occur, and the calculations may therefore easily be effected by means of a table of Differences of Nothing such as that given § 3, Chapter III. Writing the series for u_{k+x} in the form

$$u_{k+x} = u_k + \frac{x}{n}\Delta + \frac{x^2 - nx}{\lfloor 2 \, n^2}\Delta^2 + \frac{x^3 - 3nx^2 + 2n^2x}{\lfloor 3 \, n^3}\Delta^3$$

$$+ \frac{x^4 - 6nx^3 + 11n^2x^2 - 6n^3x}{\lfloor 4 \, n^4}\Delta^4 + \frac{x^5 - 10nx^4 + 35n^2x^3 - 50n^3x^2 + 24n^4x}{\lfloor 5 \, n^5}\Delta^5 +$$

where Δ, Δ^2, . . . are abreviations for Δu_k, $\Delta^2 u_k$ . . ., and making use of our table of the Differences of Nothing we write at once

$$\delta u_k = \frac{\Delta}{n} + \frac{1-n}{\underline{|2}\,n^2}\Delta^2 + \frac{1-3n+2n^2}{\underline{|3}\,n^3}\Delta^3$$

$$+ \frac{1-6n+11n^2-6n^3}{\underline{|4}\,n^4}\Delta^4 + \frac{1-10n+35n^2-50n^3+24n^4}{\underline{|5}\,n^5}\Delta^5 + \dots$$

$$\delta^2 u_k = \frac{2}{\underline{|2}\,n^2}\Delta^2 + \frac{6-6n}{\underline{|3}\,n^3}\Delta^3 + \frac{14-36n+22n^2}{\underline{|4}\,n^4}\Delta^4$$

$$+ \frac{30-140n+210n^2-100n^3}{\underline{|5}\,n^5}\Delta^5 + \dots$$

$$\delta^3 u_k = \frac{6}{\underline{|3}\,n^3}\Delta^3 + \frac{36-36n}{\underline{|4}\,n^4}\Delta^4 + \frac{150-360n+210n^2}{\underline{|5}\,n^5}\Delta^5 + \dots$$

$$\delta^4 u_k = \frac{24}{\underline{|4}\,n^4}\Delta^4 + \frac{240-240n}{\underline{|5}\,n^5}\Delta^5 + \dots$$

$$\delta^5 u_k = \frac{120}{\underline{|5}\,n^5}\Delta^5 + \dots$$

Suppose for example that $n=10$ and that six orders of differences are retained. Then

$$\delta = \cdot 1\Delta - \cdot 045\Delta^2 + \cdot 0285\Delta^3 - \cdot 0206625\Delta^4$$
$$+ \cdot 01611675\Delta^5 - \cdot 0131620125\Delta^6$$

$$\delta^2 = \cdot 01\Delta^2 - \cdot 009\Delta^3 + \cdot 007725\Delta^4 - \cdot 0066975\Delta^5$$
$$+ \cdot 005895225\Delta^6$$

$$\delta^3 = \cdot 001\Delta^3 - \cdot 00135\Delta^4 + \cdot 0014625\Delta^5 - \cdot 0014805\Delta^6$$

$$\delta^4 = \cdot 0001\Delta^4 - \cdot 00018\Delta^5 + \cdot 0002355\Delta^6$$

$$\delta^5 = \cdot 00001\Delta^5 - \cdot 0000225\Delta^6$$

$$\delta^6 = \cdot 000001\Delta^6$$

(compare *Journal of the Institute of Actuaries,* vol. xiii, p. 160).

§ 2. As an example of the problem discussed in the preceding section, we might be required to find the logarithms to base 10 of all numbers from 1,000 to 10,000, having given the logarithms of the numbers 100 to 1,000, that is the logarithms of the numbers 1,000, 1,010, 1,020 . . . 10,000. The table of differences Δ, Δ^2 . . . would be formed from the given logarithms by subtraction, the differences δ, δ^2, δ^3 . . . would then be found for each interval by means of the preceding equations, and these differences being found the quantities to be interpolated would be found by addition. The quantities Δ, Δ^2 . . . change in passing from one interval to the next, but the coefficients by which they are multiplied in the expressions for δ, δ^2 . . . of course remain unaltered throughout the calculations.

It must not be inferred that this would be the most expeditious way of making the interpolations in question. The illustration is given merely to assist the student in understanding the nature of the problem discussed in § 1.

§ 3. *Lagrange's Interpolation Formula.* Given m values of a function which are not equidistant, to find any other value of the function.

Let $u_a, u_b, u_c \ldots u_k$ be the given values, corresponding to the values $a, b, c \ldots k$ of x. It is required to find the general expression for u_x.

Assuming u_x is a rational integral algebraic function of x of degree $m-1$, it may be put in the form

$$\begin{aligned}\mathrm{A}(x-b)(x-c)\ldots(x-k)+\mathrm{B}(x-a)(x-c)\ldots(x-k)\\+\ldots+\mathrm{K}(x-a)(x-b)\ldots\end{aligned}$$

where A, B, C . . . are quantities independent of x. For this expression is of the $\overline{m-1}$th degree in x, and since it contains m independent constants A, B, C . . . K it may be made, by a proper choice of these constants, to represent any function whatever of the $\overline{m-1}$th degree in x. We have to find what values must be given to A, B, C . . . K that the expression may represent u_x.

Putting $x=a$ we see that we must have

$$u_a=\mathrm{A}(a-b)(a-c)\ldots(a-k)$$

$$\therefore\quad \mathrm{A}=\frac{u_a}{(a-b)(a-c)\ldots(a-k)}$$

Similar values are obtained for B, C . . . Thus we get

$$\begin{aligned}u_x=u_a\frac{(x-b)(x-c)\ldots(x-k)}{(a-b)(a-c)\ldots(a-k)}+u_b\frac{(x-a)(x-b)\ldots(x-k)}{(b-a)(b-c)\ldots(b-k)}\\+\ldots+u_k\frac{(x-a)(x-b)\ldots}{(k-a)(k-b)\ldots}\end{aligned}$$

This is called Lagrange's Formula for Interpolation.

§ 4. The following transformation of Lagrange's formula is of use in the theory of interpolation.

We have

$$1+\frac{x-a}{a-b}+\frac{(x-a)\,.\,(x-b)}{(a-b)\,.\,(a-c)}+\frac{(x-a)(x-b)(x-c)}{(a-b)(a-c)(a-d)}+\ldots$$

$$=\frac{x-b}{a-b}+\frac{(x-a)(x-b)}{(a-b)(a-c)}+\frac{(x-a)(x-b)(x-c)}{(a-b)(a-c)(a-d)}+\ldots$$

$$=\frac{x-b}{a-b}\left\{1+\frac{x-a}{a-c}+\frac{(x-a)(x-c)}{(a-c)(a-d)}+\ldots\right\}$$

and this expression may be shown in the same way to be equal to

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}\left\{1+\frac{x-a}{a-d}+\frac{(x-a)(x-d)}{(a-d)(a-e)}+\ldots\right\}$$

$$=\text{, finally}\qquad \frac{(x-b)(x-c)\,\ldots\,(x-k)}{(a-b)(a-c)\,\ldots\,(a-k)}$$

By means of this equation Lagrange's formula may be written

$$u_x=u_a\left\{1+\frac{x-a}{a-b}+\frac{(x-a)(x-b)}{(a-b)(a-c)}+\frac{(x-a)(x-b)(x-c)}{(a-b)(a-c)(a-d)}+\ldots\right\}$$

$$+u_b\frac{x-a}{b-a}\left\{1+\frac{x-b}{b-c}+\frac{(x-b)(x-c)}{(b-c)(b-d)}+\ldots\right\}$$

$$+u_c\frac{(x-a)(x-b)}{(c-a)(c-b)}\left\{1+\frac{x-c}{c-d}+\ldots\right\}$$

$$+\ldots\ldots$$

$$=u_a+(x-a)\left\{\frac{u_a}{a-b}+\frac{u_b}{b-a}\right\}+(x-a)(x-b)\left\{\frac{u_a}{(a-b)(a-c)}+\frac{u_b}{(b-a)(b-c)}+\frac{u_c}{(c-a)(c-b)}\right\}$$

$$+(x-a)(x-b)(x-c)\left\{\frac{u_a}{(a-b)(a-c)(a-d)}+\ldots\right\}+\ldots$$

$$+(x-a)(x-b)\,\ldots\,(x-h)\left\{\frac{u_a}{(a-b)(a-c)\,\ldots\,(a-k)}+\ldots+\frac{u_k}{(k-a)(k-b)\,\ldots\,(k-h)}\right\}$$

where k denotes the last of the quantities $a, b, c\ldots$ and h the last but one.

§ 5. As a particular case let each of the quantities $b, c, d \ldots$ exceed that which precedes it by n, *i.e.*, let $b-a=c-b= \ldots =k-h=n$. Then the expression

$$\frac{u_a}{(a-b)(a-c)\ldots}+\frac{u_b}{(b-a)(b-c)}+\ldots \quad . \quad . \quad (a)$$

in which the number of factors in each denominator is r

$$=\frac{(-1)^r}{n^r}\left\{\frac{u_a}{\underline{|r}}-\frac{u_b}{1\,.\,\underline{|r-1}}+\frac{u_c}{\underline{|2}\,\underline{|r-2}}-\ldots\right\}$$

$$=\frac{(-1)^r}{n^r\underline{|r}}\left\{u_a-\frac{r}{1}\cdot u_b+\frac{r\,.\,(r-1)}{\underline{|2}}u_c-\ldots\right\}$$

Now, the increment of x being n,

$$\Delta^r u_x = u_{x+nr}-ru_{x+\overline{r-1}.n}+\frac{r\,.\,(r-1)}{\underline{|2}}u_{x+\overline{r-2}n}-\ldots$$

$$=(-1)^r\left\{u_x-ru_{x+n}+\frac{r\,.\,(r-1)}{\underline{|2}}u_{x+2n}-\ldots\right\}$$

From which we see that the expression (a) becomes $\dfrac{\Delta^r u_a}{n^r\underline{|r}}$.

The expression (a) involves the quantities $a, b, c \ldots$ symmetrically, and we therefore see that if they can be arranged in any order $a, \beta, \gamma \ldots$ such that $\beta-a=\gamma-\beta=\ldots=n$, then it takes the value $\dfrac{\Delta^r u_a}{n^r\underline{|r}}$.

Now take $\quad a=0 \qquad b=n$

$\qquad c=-n \qquad d=2n$

$\qquad e=-2n \qquad f=3n$

$\qquad \ldots \qquad \ldots$

Then the expression for u_x becomes

$$u_x=u_0+\frac{x}{n}\Delta u_0+\frac{x(x-n)}{\underline{|2}n^2}\Delta^2 u_{-n}+\frac{x(x^2-n^2)}{\underline{|3}n^3}\Delta^3 u_{-n}$$

$$+\frac{x(x^2-n^2)(x-2n)}{\underline{|4}n^4}\Delta^4 u_{-2n}+\frac{x(x^2-n^2)(x^2-4n^2)}{\underline{|5}n^5}\Delta^5 u_{-2n}+\ldots \quad (\beta)$$

Again take $\quad a=0 \qquad b=-n$

$\qquad c=n \qquad d=-2n$

$\qquad e=2n \qquad f=-3n$

$\qquad \ldots \qquad \ldots$

Then the expression becomes

$$u_x = u_0 + \frac{x}{n}\Delta u_{-n} + \frac{x(x+n)}{\underline{|2}\,n^2}\Delta^2 u_{-n} + \frac{x(x^2-n^2)}{\underline{|3}\,n^3}\Delta^3 u_{-2n}$$

$$+ \frac{x(x^2-n^2)(x+2n)}{\underline{|4}\,n^4}\Delta^4 u_{-2n} + \frac{x(x^2-n^2)(x^2-4n^2)}{\underline{|5}\,n^5}\Delta^5 u_{-3n} + \ldots$$

Adding these two expressions for u_x we obtain Stirling's Interpolation Formula, viz.:—

$$u_x = u_0 + \frac{x}{n}\cdot\tfrac{1}{2}\Delta(u_0 + u_{-n}) + \frac{x^2}{\underline{|2}\,n^2}\Delta^2 u_{-n} + \frac{x(x^2-n^2)}{\underline{|3}\,n^3}\tfrac{1}{2}\Delta^3(u_{-n} + u_{-2n})$$

$$+ \frac{x^2(x^2-n^2)}{\underline{|4}\,n^4}\Delta^4 u_{-2n} + \frac{x(x^2-n^2)(x^2-4n^2)}{\underline{|5}\,n^5}\tfrac{1}{2}\Delta^5(u_{-2n} + u_{-3n}) + \ldots$$

§ 6. Stirling's interpolation formula gives in another form a solution of the problem considered in § 1. If we assume that the mth and higher differences vanish it may be shown that the two give identically the same values for u_x. These conditions however will not be exactly satisfied in any actual case, (except of course where the function may be represented exactly by an expression of the $\overline{m-1}$th degree in x). The results obtained from the formulæ of sections 1 and 6 and from other interpolation formulæ will therefore generally differ to a greater or less extent, and it becomes a question of importance what formula may be most advantageously employed with a view to making the calculations as short and the results as correct as possible.

In searching for an interpolation formula to subdivide any interval it seems natural to look for one which would involve the given values of the function on either side the interval symmetrically. For instance to interpolate in the interval between u_0 and u_n we should look for a formula which would involve u_0 in the same way as u_n, u_{-n} in the same way as u_{2n}, and so on. It is obvious that Stirling's formula is not strictly appropriate for this purpose. On referring to the difference table given p. 12, it will be seen that we require an equation of the form

$$u_x = \mathrm{A}(u_0 + u_n) + \mathrm{B}\Delta u_0 + \mathrm{C}\Delta^2(u_0 + u_{-n}) + \mathrm{D}\Delta^3 u_{-n} + \ \ldots$$

The quantities A, B, C, D, E . . . must be functions of x and n such that the calculations might equally well be performed in reverse order, with the series of given quantities . . . u_{-2n}, u_{-n}, u_0, u_n, u_{2n}, u_{3n} . . . reversed. Now if we reverse the order of the given quantities the signs of the odd differences Δ, Δ^3 . . . will be changed but the differences otherwise will remain unaltered. We therefore see that when x is changed into $n-x$, A, B, C, D, E . . .

must remain numerically unaltered, but the signs of the even coefficients B, D . . . must change.

A formula of this kind may be found without difficulty by the method of § 5. Referring to it and putting

$$a=n \qquad b=\ \ 0$$
$$c=2n \qquad d=\ -n$$
$$e=3n \qquad f=\ -2n \ldots \text{ we get}$$

$$u_x=u_n+\frac{(x-n)}{n}\Delta u_0+\frac{(x-n)x}{\lfloor 2 \, n^2}\Delta^2 u_0+\frac{(x-n)x(x-2n)}{\lfloor 3 \, n^3}\Delta^3 u_{-n}$$
$$+\frac{x(x^2-n^2)(x-2n)}{\lfloor 4 \, n^4}\Delta^4 u_{-n}+\frac{x(x^2-n^2)(x-2n)(x-3n)}{\lfloor 5 \, n^5}\Delta^5 u_{-2n}+\ldots$$

Adding this to equation (β) of § 5, we obtain

$$=\tfrac{1}{2}(u_0+u_n)+\frac{2x-n}{2n}\Delta u_0+\frac{x(x-n)}{\lfloor 2 \, n^2}\tfrac{1}{2}\Delta^2(u_0+u_{-n})+\frac{x(x-n)}{\lfloor 3 \, n^3}\cdot\frac{2x-n}{2}\Delta^3 u_{-n}$$
$$+\frac{x(x^2-n^2)(x-2n)}{\lfloor 4 \, n^4}\tfrac{1}{2}\Delta^4(u_{-n}+u_{-2n})+\frac{x(x^2-n^2)(x-2n)}{\lfloor 5 \, n^5}\cdot\frac{2x-n}{2}\Delta^5 u_{-2n}+\ldots$$

With reference to the interval between u_0 and u_n, the quantities Δu_0, $\tfrac{1}{2}\Delta^2(u_0+u_{-n})$, $\Delta^3 u_{-n}$, . . . are called "central differences." In any interpolation formula in which u_x is expanded in a series of differences of ascending order, the term involving the difference of any order is called the equation of that difference. For instance, in the formula just given, the term $\frac{x\,.\,\overline{x-n}}{\lfloor 2 \, n^2}\tfrac{1}{2}\Delta^2(u_0+u_{-n})$ is called the equation of the second difference.

§ 7. As an illustration of Sections (1) and (6), let us take from the *Institute of Actuaries' Text-Book,* Part I, Interest, p. 167, the values of $\log_{10}(1+i)$ for $100i=6\frac{4}{16}$, $6\frac{14}{16}$, $7\frac{8}{16}$, . . . 10, and deduce from them the values of $\log_{10}(1+i)$ for $100i=7\frac{9}{16}$, $7\frac{10}{16}$, $7\frac{11}{16}$, . . . $8\frac{1}{16}$; that is, fill up the interval between $7\frac{8}{16}$ and $8\frac{2}{16}$.

Retaining 13 figures in the logarithms, and omitting the initial zeros, we obtain by subtraction the following difference table :—

Table A.

		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
u_{-20}	263 289 387 223					
		25 471 890 139				
u_{-10}	288 761 277 362		−148 524 985			
		25 323 365 154		1 722 044		
u_0	314 084 642 516		−146 802 941		−29 778	
		25 176 562 213		1 692 266		686
u_{10}	339 261 204 729		−145 110 675		−29 092	
		25 031 451 538		1 663 174		660
u_{20}	364 292 656 267		−143 447 501		−28 432	
		24 888 004 037		1 634 742		
u_{30}	389 180 660 304		−141 812 759			
		24 746 191 278				
u_{40}	413 926 851 582					

Let us first consider the formula of § 6. Retaining fourth differences, it may be written

$$u_x = \tfrac{1}{2}(u_0+u_n) + \frac{2x-n}{2n}\Delta u_0 + \frac{x^2-nx}{\underline{|2} n^2}\tfrac{1}{2}\Delta^2(u_0+u_{-n})$$

$$+\frac{2x^3-3nx^2+n^2x}{2\underline{|3} n^3}\Delta^3 u_{-n} + \frac{x^4-2nx^3-n^2x^2+2n^3x}{\underline{|4} n^4}\tfrac{1}{2}\Delta^4(u_{-n}+u_{-2n})$$

from which, by the method of § 1, we obtain

$$\delta u_0 = \frac{\Delta u_0}{10} - \frac{9}{200}\tfrac{1}{2}\Delta^2(u_0+u_{-10}) + \frac{6}{1000}\Delta^3 u_{-10} + \frac{627}{80000}\tfrac{1}{2}\Delta^4(u_{-10}+u_{-20})$$

$$\delta^2 u_0 = \frac{1}{100}\tfrac{1}{2}\Delta^2(u_0+u_{-10}) - \frac{4}{1000}\Delta^3 u_{-10} - \frac{51}{40000}\cdot\tfrac{1}{2}\Delta^4(u_{-10}+u_{-20})$$

$$\delta^3 u_0 = \frac{\Delta^3 u_{-10}}{1000} - \frac{7}{20000}\tfrac{1}{2}\Delta^4(u_{-10}+u_{-20})$$

$$\delta^4 u_0 = \frac{1}{10000}\cdot\tfrac{1}{2}\Delta^4(u_{-10}+u_{-20}),$$

whence

$$\begin{aligned}\delta u_0 &= 2\,524\,234\,200{\cdot}559\,187\,5\\ \delta^2 u_0 &= -1\,466\,299{\cdot}614\,375\\ \delta^3 u_0 &= 1\,702{\cdot}568\,25\\ \delta^4 u_0 &= -2{\cdot}943\,5\end{aligned}$$

The above are the exact values of δu_0, $\delta^2 u_0 \ldots$ as deduced from the data. If, instead of these values, we use approximate values d_1, d_2, d_3, d_4, obtained by omitting some of the last figures of the decimal parts, and if E_1, E_2, E_3, E_4 are the errors thereby introduced into δ, δ^2, δ^3, δ^4, so that $d_1=\delta+E_1$, $d_2=\delta^2+E_2 \ldots$, the value found for u_x will be

$$u_0 + xd_1 + \frac{x(x-1)}{\underline{|2}}d_2 + \ldots + \frac{x.(x-1)(x-2)(x-3)}{\underline{|4}}d_4$$

so that the error thereby introduced into u_x will be

$$xE_1 + \frac{x.(x-1)}{\underline{|2}}E_2 + \frac{x(x-1)(x-2)}{\underline{|3}}E_3 + \frac{x(x-1)(x-2)(x-3)}{\underline{|4}}E_4.$$

The coefficients of E_1, $E_2 \ldots$ are greatest when $x=10$ when they have the values 10, 45, 120, 210. If, then, we retain two decimal places in δu_0, and three in each of the others, the greatest error introduced will be less than $10\times{\cdot}005+(45+120+210)\times{\cdot}0005={\cdot}2375$.

Let us then take the decimal parts of δ, $\delta^2 \ldots$ to be ·56, ·614,

·568 and ·943 respectively. These give us $E_1 = {\cdot}0008125$, $E_2 = {\cdot}000375$, $E_3 = -{\cdot}00025$, $E_4 = {\cdot}0005$, so that the error introduced into u_{10} is ${\cdot}008125 + {\cdot}016875 - {\cdot}03 + {\cdot}105 = {\cdot}1$, and the maximum error introduced is less than ${\cdot}008125 + {\cdot}016875 + {\cdot}03 + {\cdot}105 = {\cdot}16$.

We now proceed to the interpolations. $\delta^3 u_0$ is first written down (see Table B), and then $\delta^3 u_1$, $\delta^3 u_2 \ldots$ formed from it by repeated addition of $\delta^4 u = \bar{1}7{\cdot}057$. These are written on every fifth line. The value of $\delta^2 u_0$ is then written above $\delta^3 u_0$, the two added together, and the sum—*i.e.*, $\delta^2 u_1$—written above $\delta^3 u_1$, then $\delta^3 u_1$ and $\delta^2 u_1$ are added, and the sum—*i.e.*, $\delta^2 u_2$—written above $\delta^3 u_2$, and so on. In this way the second differences are formed, and then by a similar process the first differences, and finally the quantities $u_1, u_2, \ldots u_{10}$.

Table B.

u_0	314 084 642 516	u_5	326 691 167 533·625
δ	2 524 234 200·56		2 516 919 698·740
δ_2	$\bar{1}$8 533 700·386		$\bar{1}$8 542 183·796
δ_3	1 702·568		1 687·853
u_1	316 608 876 716·56	u_6	329 208 087 232·365
	2 522 767 900·946		2 515 461 882·536
	$\bar{1}$8 535 402·954		$\bar{1}$8 543 871·649
	1 699·625		1 684·910
u_2	319 131 644 617·506	u_7	331 723 549 114·901
	2 521 303 303·900		2 514 005 754·185
	$\bar{1}$8 537 102·579		$\bar{1}$8 545 556·559
	1 696·682		1 681·967
u_3	321 652 947 921·406	u_8	334 237 554 869·086
	2 519 840 406·479		2 512 551 310·744
	$\bar{1}$8 538 799·261		$\bar{1}$8 547 238·526
	1 693·739		1 679·024
u_4	324 172 788 327·885	u_9	336 750 106 179·830
	2 518 379 205·740		2 511 098 549·270
	$\bar{1}$8 540 493·000		$\bar{1}$8 548 917·550
	1 690·796		1 676·081
		u_{10}	339 261 204 729·100

If we go back to the formula (4) of § 6, we notice that when $x = n$ the series takes the value u_n whatever values we assign to $\Delta^2(u_0 + u_{-n})$, $\Delta^3 u_{-n} \ldots$; and therefore from the fact that u_{10} comes out correct, except for the predicted error ·1, we may not infer that the calculations are free from error, though we may infer that the additions of Table B have probably been correctly performed.

§ 8. The method in which the computations involved in any interpolation may most advantageously be effected will depend on the particular problem under consideration. Two points may be noticed.

(1). The equations of the differences diminish as the order of the differences increases.

(2). That when, as in § 7, the calculations are effected by continued addition of differences, there is an accumulation of error, which frequently increases with the order of the difference. For instance, in § 7, an error in $\delta^1 u_0$ is magnified 10 times in u_{10}, while an error in $\delta^4 u_0$ is magnified 210 times in u_{10}. Owing to this it is often necessary to retain in the higher differences several figures more than are wanted in the final results.

For these, among other reasons, it is sometimes profitable, not to proceed as in § 7, but to introduce the equations of the higher differences into the calculations by a different process.

Let us now make use of the formula of § (1) to perform the interpolations already effected in Table B, first obtaining approximate values of the quantities to be interpolated by proceeding as in § 7, but retaining only second differences, and then correcting the results so obtained by adding the equations of the higher differences.

Retaining only second differences, the formulæ of § 1 give us

$$\delta u_0 = \frac{\Delta u_0}{10} - \cdot 045 \Delta^2 u_0 = 2\,524\,186\,201\cdot 675$$

$$\delta^2 u_0 = \cdot 01 \Delta^2 u_0 \qquad = -\,145\,110\,6\cdot 75$$

$$= \bar{1}\,854\,889\,3\cdot 25$$

The Table C (given below) is formed as follows:—

$\delta u_0 = 2\,524\,186\,201\cdot 675$ is first written down at the top of the table, then by repeated addition of δ^2 we form δu_1, $\delta u_2 \ldots$, written on every seventh line. u_0 is then written underneath δu_0, the two added, and the sum written underneath δu_1, and so on.

The results so obtained require to be corrected by the equations of the third, fourth and fifth differences, which have been obtained from their values

$$\frac{x(x-10)(x-20)}{\lfloor 3 (10)^3} \times 1\,663\,174$$

$$-\frac{x(x-10)(x-20)(x-30)}{\lfloor 4 (10)^4} \times 28\,432$$

$$\frac{x(x-10)(x-20)(x-30)(x-40)}{\lfloor 5 (10)^5} \times 640$$

by direct calculation. These are written in order underneath, and each of the final results u_1, u_2 . . . is then obtained by addition of the four numbers above it. We notice that $\Delta^5 u_0$ cannot be obtained exactly from our data, but on the assumption that sixth differences are constant, its value is 634. It has been put = 640, which is sufficiently exact for our purpose.

Table C.

	2 524 186 201·675
u_0	314 084 642 516
	2 522 735 094·925
	316 608 828 717·675
	47 400·46
	587·48
	10·3
u_1	316 608 876 715·9
	2 521 283 988·175
	319 131 563 812·600
	79 832·35
	955·31
	16·4
u_2	319 131 644 616·7
	2 519 832 881·425
	321 652 847 800·775
	98 958·85
	1 141·90
	19
u_3	321 652 947 920·5
	2 518 381 774·675
	324 172 680 682·200
	106 443·14
	1 182·77
	19·2
u_4	324 172 788 327·3
	2 516 930 667·925
	326 691 062 456·875
	103 948·38
	1 110·62
	17·5
u_5	326 691 167 533·4
	2 515 479 561·175
	329 207 993 124·800
	93 137·74
	955·32
	14·6
u_6	329 208 087 232·5
	2 514 028 454·425
	331 723 472 685·975
	75 674·42
	743·85
	11
u_7	331 723 549 115·2
	2 512 577 347·675
	334 237 501 140·400
	53 221·57
	500·40
	7·2
u_8	334 237 554 869·6
	2 511 126 240·925
	336 750 078 488·075
	27 442·37
	246·29
	3·4
u_9	336 750 106 180·1
	2 509 675 134·175
u_{10}	339 261 204 729·000

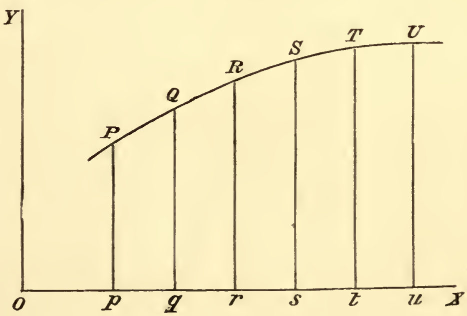
§ 9. When a single interval only has to be filled up, it will generally, as in the example we have considered, be far easier to calculate all the leading differences, and then form the table by addition, as in § 7. But if a large series of intervals have to be filled up, the processes may sometimes be shortened either by computing the equations of the higher differences directly or by the use of various artifices to introduce them into the calculations, subsidiary tables being formed to shorten the work. (*Vide* paper on Interpolation by Mr. Woolhouse, *Journal of the Institute of Actuaries*, vol. xi, p. 61.)

The figures of Table C afford a useful illustration of the gradual closing of the approximations as the successive differences are taken account of, and it is for this reason that the table has been given in the shape in which it stands.

§ 10. On comparing the interpolated quantities found in §§ 7 and 8 with the values of the quantities as given on p. 167 of the *Institute Text Book*, it will be seen that the results of each interpolation are correct, or nearly so, to the last figure retained, the greatest error being about ·5 in this figure. If we calculate for Table C the equations of the 6th differences, we shall find the greatest of them is about ·7. For the formula used in § 7, the greatest value of the equation of the 5th difference is about ·6. We therefore see that the results obtained by the use of formula (4), retaining only 4th differences, are at least as accurate as those given by formula (3) retaining 5th differences. It is easily seen that the series (4) generally converges more rapidly than the series (3).

* § 11. It is not to be supposed that all interpolation formulæ are founded on the assumption of § 2, Chap. II. Mr. Sprague, in a very interesting paper (*Journal of the Institute of Actuaries*, vol. xxii, p. 270) has given formulæ for interpolation framed on a different basis, which deserve attentive study. The hypothesis on which he proceeds may be explained as follows:—

Let the ordinates Pp, Qq . . . represent given values y_0, y_1 . . . of a function between which it is desired to interpolate other values. "The problem ov "interpolating "between y_2 and "y_3 is the same "as that ov drawing a curvd line between the points R and S,



" and in order to get a satisfactory interpolation, it is necessary " that this partial curv shoud join on smoothly to the ajacent " partial curvs, namely, QR on the one side and ST on the " other side."

In the formulæ for interpolation hitherto considered, this smoothness of junction is obtained, with more or less success, by drawing the partial curves so that any one, as RS, forms a portion of a curve through R, S, and adjacent given points. This partial curve approximates to a portion of the parabolic curve of degree $m-1$ passing through all the given points, supposed m in number, coinciding exactly with this curve when $m-1$ orders of differences are retained.

Mr. Sprague secures this smoothness of junction by drawing the partial curves so that any two adjacent ones, as QR, RS, have, at their common point R, contact of the second order with the quartic parabola passing through R and the two adjacent given points on each side, PQ and ST. Each partial curve has therefore to satisfy three conditions at each extremity, six in all.

Let us find the equation to the curve RS, on the assumption that the given ordinates are equidistant. Taking r as origin, it will be of the form

$$y=y_2+ax+bx^2+cx^3+dx^4+ex^5 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{I}$$

Now let us form the table of differences for the quantities $y_0, y_1, y_2 \ldots y_5$. Denote the differences of y_0 by $\Delta, \Delta^2, \Delta^3 \ldots$ and those of y_1 by $\Delta_1, \Delta_1^2, \Delta_1^3 \ldots$

Referred to p as origin, the equation of the quartic parabola through P, Q, R, S, T, is

$$y=y_0+x\Delta+\frac{x(x-1)}{\underline{|2}}\Delta^2+\frac{x(x\ \ 1)(x-2)}{\underline{|3}}\Delta^3$$
$$+\frac{x(x-1)(x-2)(x-3)}{\underline{|4}}\Delta^4$$

Transferring the origin to r, this becomes

$$y=y_0+(x+2)\Delta+\frac{(x+2)(x+1)}{\underline{|2}}\Delta^2+\frac{(x+2)(x+1)x}{\underline{|3}}\Delta^3$$
$$+\frac{(x+2)(x+1)x(x-1)}{\underline{|4}}\Delta^4$$

$$=y_2+x(\Delta+\tfrac{3}{2}\Delta^2+\tfrac{1}{3}\Delta^3-\tfrac{1}{12}\Delta^4)+x^2(\tfrac{1}{2}\Delta^2+\tfrac{1}{2}\Delta^3-\tfrac{1}{24}\Delta^4)+\ldots \quad \text{II}$$

The equation I, referred to s as origin, is

$$y=y_2+a+b+c+d+e+x\{a+2b+3c+4d+5e\}$$
$$+x^2\{b+3c+6d+10e\}+\ldots \quad \text{III}$$

and, by observing the form of Equation II, we see that, referred to s as origin, the equation to the quartic parabola through $QRSTU$ is

$$y=y_3+x(\Delta_1+\tfrac{3}{2}\Delta_1{}^2+\tfrac{1}{3}\Delta_1{}^3-\tfrac{1}{12}\Delta_1{}^4)+x^2(\tfrac{1}{2}\Delta_1{}^2+\tfrac{1}{2}\Delta_1{}^3-\tfrac{1}{24}\Delta_1{}^4)+\ldots \text{IV}$$

Since I and II have contact of the second order at R, we obtain, equating constant terms and the coefficients of x and x^2,

$$a=\Delta+\tfrac{3}{2}\Delta^2+\tfrac{1}{3}\Delta^3-\tfrac{1}{12}\Delta^4 \quad \ldots \quad (\alpha)$$
$$b=\tfrac{1}{2}\Delta^2+\tfrac{1}{2}\Delta^3-\tfrac{1}{24}\Delta^4 \quad \ldots \quad (\beta)$$

Similarly, since III and IV have contact of the second order at S,

$$a+b+c+d+e=y_3-y_2=\Delta+2\Delta^2+\Delta^3 \quad \ldots \quad (\gamma)$$
$$a+2b+3c+4d+5e=\Delta_1+\tfrac{3}{2}\Delta_1{}^2+\tfrac{1}{3}\Delta_1{}^3-\tfrac{1}{12}\Delta_1{}^4$$
$$=\Delta+\tfrac{5}{2}\Delta^2+\tfrac{11}{6}\Delta^3+\tfrac{1}{4}\Delta^4-\tfrac{1}{12}\Delta^5 \quad \ldots \quad (\delta)$$
$$b+3c+6d+10e=\tfrac{1}{2}\Delta_1{}^2+\tfrac{1}{2}\Delta_1{}^3-\tfrac{1}{24}\Delta^4$$
$$=\tfrac{1}{2}\Delta^2+\Delta^3+\tfrac{11}{24}\Delta^4-\tfrac{1}{24}\Delta^5 \quad \ldots \quad (t)$$

The five equations (α), (β), (γ), (δ), (t), which are the same as those obtained by Mr. Sprague by a somewhat different process, determine a, b, c, d, e, in terms of Δ, $\Delta^2 \ldots \Delta^5$. Solving them, and substituting in Equation I, we get the required interpolation formula.

EXERCISES AND EXAMPLES.

1. Show, by actual addition, the succeeding terms of a series the first of which is u and the successive differences of whose terms are Δu, $\Delta^2 u$, $\Delta^3 u$ . . . .

"Institute of Actuaries' Exam.", vide *Journal* iii, p. 274.

2. Point out the analogies which lead us to infer that for the nth term we may write $u_n = u + n\Delta u + \frac{n.\overline{n-1}}{1.2}\Delta^2 u + \ldots$ and for the nth difference $\Delta^n u = u_n - nu_{n-1} + \frac{n.\overline{n-1}}{1.2} u_{n-2} = \ldots.$

"Institute of Actuaries' Exam.", vide *Journal* iii, p. 274.

3. Find $\delta^n a^x$ when x is variable, the increment of x being unity.

"Institute of Actuaries' Exam.", vide *Journal* xviii, p. 378.

4. If u_x be any function of x of n dimensions, prove that $\Delta^n u_x$ is constant; and hence show how to form a table of cubes of natural numbers expeditiously.

"Institute of Actuaries' Exam.", vide *Journal* xxii, p. 65.

5. Investigate the expressions:

(a) for u_{x+n} in terms of Δu_x.

(β) for $\Delta^n u_x$ in terms of u_x and its successive values.

"Institute of Actuaries' Exam.", vide *Journal* xxii, p. 65.

6. Investigate an expression for $\Delta^n u_x$ in terms of u_x and its successive values.

Using the formula thus found, if in the series 1, 6, 21, 56, κ, 252, 462, &c., the sixth differences vanish, find κ and sum the series to 10 terms.

"Institute of Actuaries' Exam.", vide *Journal* xxi, p. 224.

7. Given $u_x = 100{,}000$, $\delta_1 = -490$, $\delta_2 = 93$, $\delta_3 = -25$, $\delta_4 = 2$, $\delta_5 = 0$, find the first eight terms of the series u_x, u_{x+1}, &c.

"Institute of Actuaries' Exam.", vide *Journal* xxii, p. 65.

8. Find u_{12} and also u_2 when $u_5 = 55$, $u_6 = 126$, $u_7 = 259$, $u_8 = 484$, $u_9 = 837$, and Δ^4 is constant.

"Institute of Actuaries' Exam.", 1881.

9. Given the following values, construct logs 7·1, 7·2 . . . 7·9, 8, to seven places, and explain why the last differs from log 8 as obtained from an ordinary table of logs.

$$\begin{aligned} \log 7 &= \cdot 8450980 \\ \delta_1 &= 61603 \\ \delta_2 &= -861 \\ \delta_3 &= 24 \\ \delta_4 &= -1 \end{aligned}$$

"Institute of Actuaries' Exam.", vide *Journal* xxi, p. 224.

10. Given

$$\begin{aligned} \log 235 &= 2{\cdot}3710679 \\ \log 236 &= 2{\cdot}3729120 \\ \log 237 &= 2{\cdot}3747483 \\ \log 238 &= 2{\cdot}3765770 \end{aligned}$$

Find log 23563.

"Institute of Actuaries' Exam.", vide *Journal* xviii, p. 378.

11. Find log 512, given that

$$\begin{aligned} \log 510 &= 2{\cdot}70757018 \\ \log 511 &= 2{\cdot}70842090 \\ \log 513 &= 2{\cdot}71011737 \\ \log 514 &= 2{\cdot}71096312 \end{aligned}$$

"Institute of Actuaries' Exam.", vide *Journal* xx, p. 139.

12. Having given

$$\begin{aligned} \lambda 101 &= 2{\cdot}0043214 \\ \lambda 101{\cdot}5 &= 2{\cdot}0064660 \\ \lambda 102 &= 2{\cdot}0086002 \\ \lambda 102{\cdot}5 &= 2{\cdot}0107239 \\ \lambda 103 &= 2{\cdot}0128372 \\ \lambda 104 &= 2{\cdot}0170333 \end{aligned}$$

and assuming that sixth differences vanish find $\lambda 103{\cdot}5$.

"Institute of Actuaries' Exam.", vide *Journal* xxii, p. 65.

13. Having given the values of annuities at the following rates of interest, namely, at

$$
\begin{aligned}
3\ &= 15{\cdot}863 &&= u_0\\
3\tfrac{1}{2} &= 14{\cdot}941 &&= u_1\\
4\ &= 14{\cdot}105 &&= u_2\\
4\tfrac{1}{2} &= 13{\cdot}343 &&= u_3\\
5\ &= 12{\cdot}648 &&= u_4
\end{aligned}
$$

Find the value at 4·328 per-cent.

"Institute of Actuaries' Exam.", vide *Journal* xxii, p. 300.

14. The H[M] premium at age 40 is at 3 per-cent ·025891

„	„	„	$3\frac{1}{2}$	„	·024654
„	„	„	4	„	·023517
„	„	„	$4\frac{1}{2}$	„	·022470
„	„	„	5	„	·021509
„	„	„	6	„	·019811

Interpolate the corresponding premium at $5\frac{1}{2}$ per-cent,—(1) using *two* of these values, (2) using *four*, and (3) using *six*.

"Institute of Actuaries' Exam.", 1881.

15. Show that the sum of the series

$$\frac{1}{m(m+r)} + \frac{1}{(m+r)\,(m+2r)} + \frac{1}{(m+2r)\,(m+3r)} + \ldots \textit{ ad infin.}$$

is equal to $\dfrac{1}{mr}$.

"Institute of Actuaries' Exam." vide *Journal* iii, p 274.

16. Given every nth term of a series of values, *i.e.*, u_x, u_{x+n}, u_{x+2n}, &c., show at length how the intermediate terms, u_{x+1}, $u_{x+2}\ldots$ may be obtained by interpolation.

Given that in the series u_x, u_{x+1}, $u_{x+2}\ldots$

$$
\begin{aligned}
u_x &= 9936675{\cdot}4\\
\delta_1 &= +12767{\cdot}62\\
\delta_2 &= -3013{\cdot}725\\
\delta_3 &= +422{\cdot}8247\\
\delta_4 &= -34{\cdot}72847\\
\delta_5 &= +1{\cdot}254221
\end{aligned}
$$

you are required to construct the series as far as the term u_{x+10}. What assumption is necessary?

"Institute of Actuaries' Exam.", 1876, vide *Journal* xx, p. 139.

17. If u_x be a function of x of the form $b_1x+b_2x^2+$&c. *ad infin.*, show that it can also be expressed in the form

$$\frac{b_1x}{1-x}+\frac{\Delta b_1x^2}{(1-x)^2}+\frac{\Delta^2 b_1x^3}{(1-x)^3}+\ldots.$$

We have

$$b_{m+r+1}=b_1+(m+r)\Delta b_1+\ldots+\frac{(m+r)(m+r-1)\ldots(r+1)}{\underline{|m}}\Delta^m b_1+\ldots$$

so that the coefficient of $\Delta^m b_1$ in the term $b_{m+r+1}x^{m+r+1}$ is

$$\frac{(m+r)(m+r-1)\ldots(r+1)}{\underline{|m}}x^{m+r+1}$$

$$=x^{m+1}\frac{(m+1)(m+2)\ldots(m+r)}{\underline{|r}}x^r$$

$$=x^{m+1}\mathrm{P}_r$$

where P_r is the $(r+1)$th term in the expansion of $(1-x)^{-(m+1)}$. Thus the coefficient of $\Delta^m b_1$ in u_x is $\left(\frac{x}{1-x}\right)^{m+1}$.

18. Show that

$$u_n=\{u_{n-1}+\Delta^1 u_{n-2}+\Delta^2 u_{n-3}+\ldots+\Delta^{n-2}u_1\}+\Delta^{n-1}u_1,$$

and hence determine a series of such a nature that the terms after the first shall be respectively double the first terms of the successive orders of difference, ($u_2=2\Delta^1 u_1$, $u_3=2\Delta^2 u_1$, and so on).

"Institute of Actuaries' Exam.", 1881.

19. Prove by means of Stirling's interpolation formula that, if third and higher differences are neglected,

$$u_{-x}=\frac{x(n+x)}{2n^2}u_{-n}+\frac{n^2-x^2}{n^2}u_0-\frac{x(n-x)}{2n^2}u_n.$$

Journal of the Institute of Actuaries, xv, p. 392.

20. If $u_4=85$, $u_5=156$, $u_6=259$, $u_7=400$, $u_8=585$, and fourth differences are constant, find

(i) The differences of u_4.

(ii) The value of u_9.

(iii) The general expression for u_x.

21. Given $u_0=u_3=0$, $u_4=112$, $u_{10}=9{,}100$, $u_{20}=156{,}400$, and that fourth differences are constant, find by means of Lagrange's formula the general expression for u_x.

Result, $u_x=x^4-9x^2$.

22. Let $u_1, u_2 \ldots$ denote a series of quantities, and S_x denote the sum of the first x of them, S_0 being $=0$. Having given the values of $S_n, S_{2n} \ldots S_{rn}$, show how to find the values of $u_1, u_2, u_3 \ldots u_{rn}$.

Let δ denote the difference symbol for the increment unity in the suffix, so that $\delta u_x=u_{x+1}-u_x$, $\delta S_x=S_{x+1}-S_x=u_{x+1}$. Let Δ be the difference symbol for the increment n in the suffix, so that $\Delta u_x=u_{x+n}-u_x$, $\Delta S_x=S_{x+n}-S_x$. Form by subtraction the difference table for the series of given quantities $S_0, S_n \ldots S_{rn}$, and let $\Delta(=S_n), \Delta^2 \ldots \Delta^r$ denote the leading differences. By the method of § 1, Chap. IV, we can find δS_0, $\delta^2 S_0$, $\delta^3 S_0 \ldots \delta^r S_0$ in terms of $\Delta, \Delta^2 \ldots$. But $\delta S_x=u_{x+1}$, $\therefore \delta^2 S_x=\delta u_{x+1}$, &c., so that we obtain the values of $u_1, \delta u_1, \delta^2 u_1 \ldots \delta^{r-1}u_1$, and then by addition of differences we can form the table $u_1, u_2 \ldots u_{rn}$.

As an illustration take that afforded by Mr. Berridge's graduation of the Peerage Mortality Table (vide *Journal of the Institute of Actuaries*, vol. xii, pp. 220, 221). Here $n=10$, and the difference table for the quantities $S_0, S_n, S_{2n} \ldots$ is that given at the top of p. 221 of the *Journal*. $\Delta=S_n=99{,}616{,}210$, $\Delta^2=-18{,}425$, $\Delta^3=-9{,}898$, $\Delta^4=-186{,}096$, &c. By the equations of § 1, Chap. IV, we have

$$u_1=\cdot 1\Delta-\cdot 045\Delta^2+\cdot 0285\Delta^3-\ldots$$

$$\delta u_1=\cdot 01\Delta^2-\cdot 009\Delta^3+\ldots$$

$$\delta^2 u_1=\cdot 001\Delta^3-\ldots$$

&c. &c.

The first term, u_1, is obtained by Mr. Berridge as follows:—

$$S_n=S_0+n\delta S_0+\frac{n(n-1)}{1.2}\delta^2 S_0+\ldots$$

$$=nu_1+\frac{n(n-1)}{\underline{|2}}\delta u_1+\frac{n(n-1)(n-2)}{\underline{|3}}\delta^2 u_1+\ldots$$

$$\therefore \quad u_1=\frac{1}{n}\left\{S_n-\left(\frac{n(n-1)}{\underline{|2}}\delta u_1+\frac{n(n-1)(n-2)}{\underline{|3}}\delta^2 u_1+\ldots\right)\right\}$$

This equation gives u_1 when $\delta u_1, \delta^2 u_1 \ldots$ are found.

As another illustration, find u_1, u_2, u_3 . . ., having given $S_5=1,365$, $S_{10}=5,155$, $S_{15}=13,370$, $S_{20}=28,635$, third differences being assumed constant. It will be found that $u_1=154$, $\delta u_1=49$, $\delta^2 u_1=10$, $\delta^3 u_1=1$.

23. If $u_x=(x-a)(x-b)(x-c)$ . . . to n factors, prove that $\delta u_x=S_1+S_2+\ldots+S_n$, where, for all values of r from $r=1$ to $r=n-1$, S_r denotes the sum of the different products that can be formed from the factors $x-a$, $x-b$ . . . taken $n-r$ together, and $S_n=1$. (The increment of x is supposed unity.)

24. If $\dfrac{x(x-1)\ldots(x-n+1)}{\lfloor n}$ is denoted by $F(x, n)$, prove that, the increment of x being unity,

$$\delta^r F(x, m+r)=F(x, m).$$

25. The increment of x being unity, we have

$$u_x=u_0+x\delta u_0+\frac{x(x-1)}{\lfloor 2}\delta^2 u_0+\ldots$$

and therefore, taking $\delta^r u_x$ as the function of x,

$$\delta^r u_x=\delta^r u_0+x\delta^{r+1}u_0+\frac{x(x-1)}{\lfloor 2}\delta^{r+2}u_0+\ldots.$$

Verify this by differencing the expression

$$u_0+x\delta u_0+\frac{x(x-1)}{\lfloor 2}\delta^2 u_0+\ldots$$

and making use of Example 24.

26. Having given $u_0=a$, $\delta u_{-1}=b$, $\delta^2 u_{-2}=c$, $\delta^3 u_{-3}=d$, $\delta^4 u_{-4}=e$, prove that

$$u_x=a+xb+\frac{x(x+1)}{\lfloor 2}c+\frac{x(x+1)(x+2)}{\lfloor 3}d+\frac{x(x+1)(x+2)(x+3)}{\lfloor 4}e,$$

fourth differences being constant and the increment of x unity.

27. The increment of x being unity, prove by induction or otherwise that

$$u_x=u_0+x\delta u_{-1}+\frac{x(x+1)}{\lfloor 2}\delta^2 u_{-2}+\frac{x(x+1)(x+2)}{\lfloor 3}\delta^3 u_{-3}+\ldots.$$

Assume this theorem holds for a particular value of x. Increasing the suffixes by unity, our hypothesis gives us

$$u_{x+1} = u_1 + x\delta u_0 + \frac{x(x+1)}{\underline{|2}}\delta^2 u_{-1} + \ldots.$$

Now,
$$\begin{aligned} u_1 &= u_0 + \delta u_{-1} + \delta^2 u_{-2} + \ldots \\ \delta u_0 &= \delta u_{-1} + \delta^2 u_{-2} + \ldots \\ \delta^2 u_{-1} &= \delta^2 u_{-2} + \ldots, \text{ \&c.} \end{aligned}$$

whence the coefficient of $\delta^r u_{-r}$ in the expression for u_{x+1} is

$$1 + x + \frac{x(x+1)}{\underline{|2}} + \frac{x(x+1)(x+2)}{\underline{|3}} + \ldots + \frac{x(x+1)\ldots(x+r-1)}{\underline{|r}},$$

and it may easily be proved by induction that this

$$= \frac{(x+1)(x+2)\ldots(x+1+r-1)}{\underline{|r}}.$$

* NOTE.—The above theorem follows at once by the method of separation of symbols. Its symbolical expression is $(1+\delta)^x u_0 = \left(1 - \frac{\delta}{1+\delta}\right)^{-x} u_0$.

* **28.** Prove Briggs' Interpolation Equations (vide *Journal of the Institute of Actuaries*, vol. xiv, p. 79).

Let δ denote the difference symbol when the increment of x is unity, Δ when the increment of x is 5. Also let $1+\delta$ be denoted by E. It may be easily verified that

$$\frac{\Delta}{5} = \delta E^2 + \delta^3 E + \cdot 2\delta^5 \dagger$$

whence
$$\left(\frac{\Delta}{5}\right)^n u_x = (\delta E^2 + \delta^3 E + \cdot 2\delta^5)^n u_x.$$

† This is a particular case of the theorem

$$(1+\delta)^n - 1 = \delta^n + n\delta^{n-2}(1+\delta) + \ldots + \frac{n(n-r-1)\ldots(n-2r+1)}{\underline{|r}}\delta^{n-2r}(1+\delta)^r + \ldots + n\delta(1+\delta)^m,$$

n being an odd number $=2m+1$. This theorem was communicated to me by Mr. W. L. Mollison, of Clare Coll., Camb. It may be obtained by equating the coefficients of x^n in the equation

$$\log(1-px) + \log(1-qx) = \log\{1-(p+q)x+pqx^2\},$$

and then putting $p=1+\delta$, $q=-1$.

In this equation, putting n in succession equal to $1, 2, 3 \ldots 20$, we form the table given (*Journal of the Institute of Actuaries*, xiv, p. 79)—*e.g.*, putting $n=13$, we get

$$\left(\frac{\Delta}{5}\right)^{13}=\delta^{13}E^{26}+13\delta^{15}E^{25}+80{\cdot}6\delta^{17}E^{24}+317{\cdot}2\delta^{19}E^{23}+\ldots$$

whence

$$\delta^{13}u_{x+26}=\frac{\Delta^{13}u_x}{5^{13}}-\{13\delta^{15}u_{x+25}+80{\cdot}6\delta^{17}u_{x+24}+317{\cdot}2\delta^{19}u_{x+23}+\ldots\}.$$

Writing $\{\delta E^2+\delta^3E+{\cdot}2\delta^5\}^n$ in the form $\delta^{5n}\{a^2+a+{\cdot}2\}^n$, where $a=\frac{E}{\delta^2}$, we see that it may be expanded in a series of terms, in each of which the index of δ is greater by 2 and the index of E less by 1 than in the preceding term, the first term being δ^nE^{2n}.

Now, let two tables of differences be formed, as at p. 12, one of the δ differences, the other of the Δ differences, and imagine the second superposed on the first, the two tables being so formed that corresponding values of u_x may coincide. Since the interval between two consecutive terms of the second table is 5 times as great as between two of the first table, it follows that Δ^nu_x will fall in the $5n$th row of the first table below u_x. But $\delta^nE^{2n}u_x=\delta^nu_{x+2n}$, and is therefore in the $(4n+n)$th row below u_x—*i.e.*, in the same row in which Δ^nu_x will fall. And, since in the expansion of $\{\delta E^2+\delta^3E+{\cdot}2\delta^5\}^n$ the powers of δ increase by 2 and those of E diminish by 1, all the terms of $\{\delta E^2+\delta^3E+{\cdot}2\delta^5\}^nu_x$ will lie in the same row.

29. Denoting u_{xy}, a function of two independent variables x and y, by $\overline{xy}$, prove that, if third and higher differences are neglected,

$$\overline{xy}=\frac{1}{4}\{\overline{00}+\overline{01}+\overline{10}+\overline{11}\}+\frac{2x-1}{4}\{\overline{11}+\overline{10}-\overline{01}-\overline{00}\}$$

$$+\frac{2y-1}{4}\{\overline{11}+\overline{01}-\overline{10}-\overline{00}\}+\frac{x(x-1)}{\underline{|2}}\,\frac{1}{4}\{\overline{21}+\overline{20}+\overline{-11}+\overline{-10}$$

$$-(\overline{11}+\overline{10}+\overline{01}+\overline{00})\}+\frac{y(y-1)}{\underline{|2}}\cdot\frac{1}{4}\{\overline{12}+\overline{02}+\overline{1-1}+\overline{0-1}$$

$$-(\overline{11}+\overline{10}+\overline{01}+\overline{00})\}+\frac{2x-1}{2}\cdot\frac{2y-1}{2}\{\overline{11}+\overline{00}-\overline{01}-\overline{10}\}.$$

30. Find $\frac{1}{41 \cdot 2}$ {mantissa of $\log_{10} 41 \cdot 6$}, having given the following table of values of the function $\frac{1}{y}$ {mantissa of $\log_{10} x$}.

$x =$	$y = 40$	41	42	43
40	...	1,468,439	1,433,476	...
41	1,531,960	1,494,595	1,459,009	1,425,079
42	1,558,123	1,520,120	1,483,927	1,449,417
43	...	1,545,045	1,508,258	...

31. From the following table of endowment assurance premiums,

	50	55	60	65
15	...	£2 8 9	£2 4 6	...
20	£3 6 0	2 17 6	2 11 8	£2 7 8
25	3 18 4	3 6 0	2 17 11	2 12 6
30	...	3 19 5	3 7 8	...

in which the ages at entry are given on the left side, and the ages at which the endowments are payable at the top, find the premium for age at entry 21, sum assured payable at 58 or previous death.

Find 21 at 50, 21 at 55, 21 at 60 & 21 at 65 and then get 21 at 58.

32. Prove by means of Chap. III, § 2, or otherwise, that if r is a possible integer

$$1^r + 2^r + \ldots + n^r = \frac{n^{r+1}}{r+1}(1+a),$$

where a vanishes when n is infinite.